

A CONSTRUCTION FOR COISOTROPIC SUBALGEBRAS OF LIE BIALGEBRAS

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ABSTRACT. Given a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, we present an explicit procedure to construct coisotropic subalgebras, i.e. Lie subalgebras of \mathfrak{g} whose annihilator is a Lie subalgebra of \mathfrak{g}^* . We write down families of examples for the case that \mathfrak{g} is a classical complex simple Lie algebra.

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1. INTRODUCTION

A *Lie bialgebra* [6] structure on a Lie algebra $(\mathfrak{g}, [\bullet, \bullet])$ is a degree 1 derivation δ of $\wedge^\bullet \mathfrak{g}$ which squares to zero and satisfies $\delta([X, Y]) = [\delta(X), Y] + [X, \delta(Y)]$. Dualizing $\delta|_{\mathfrak{g}}: \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ one obtains a Lie bracket on \mathfrak{g}^* , encoding δ , so that the Lie algebra structures on \mathfrak{g} and \mathfrak{g}^* are compatible. The aim of this paper is to construct Lie subalgebras \mathfrak{h} of \mathfrak{g} with the property that \mathfrak{h}° , the subspace of \mathfrak{g}^* consisting of elements that vanish on \mathfrak{h} , is a Lie subalgebra of \mathfrak{g}^* . Such an \mathfrak{h} is called *coisotropic subalgebra*.

Our main result (Thm. 3.3) is a explicit and computationally friendly construction that works for Lie bialgebras arising from r -matrices. Recall that any r -matrix on a Lie algebra \mathfrak{g} , i.e. any $\pi \in \wedge^2 \mathfrak{g}$ such that $[\pi, \pi]$ is *ad*-invariant, gives rise to a Lie bialgebra by setting $\delta = [\pi, \bullet]$. Our result can be phrased as follows:

Theorem. *Let \mathfrak{g} be a Lie bialgebra arising from an r -matrix π . Suppose $X \in \mathfrak{g}$ satisfies*

$$[X, [X, \pi]] = \lambda[X, \pi] \text{ for some } \lambda \in \mathbb{R}.$$

Then the image of the map $\mathfrak{g}^ \rightarrow \mathfrak{g}$ given by contraction with $[X, \pi] \in \wedge^2 \mathfrak{g}$ is a coisotropic subalgebra of \mathfrak{g} .*

We remark that the coisotropic subalgebras that arise as in the theorem are all even dimensional, therefore they are by no means all coisotropic subalgebras. Using this theorem we produce in a straightforward way families of coisotropic subalgebras when \mathfrak{g} is one of the four classical simple complex Lie algebras or one of their split real forms.

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Coisotropic subalgebras give rise to lagrangian subalgebras of the Drinfeld double $\mathfrak{g} \oplus \mathfrak{g}^*$ (hence also to Poisson homogeneous spaces [8]) via $\mathfrak{k} \mapsto \mathfrak{k} \oplus \mathfrak{k}^\circ$. $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g}^*)$, the variety of lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}^*$, can be endowed with a Poisson structure [9]. It would be interesting to characterize the points of $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g}^*)$ which correspond to the coisotropic subalgebras we constructed. Notice that $\mathfrak{g} \oplus \mathfrak{g}^*$ is isomorphic to the direct sum Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ studied in [10] (see Remark 4.1). A further reason why coisotropic subalgebras are interesting is that they have a counterpart in the Hopf algebra setting after quantization [5].

Even though the above theorem is phrased entirely in terms of the Lie bialgebra \mathfrak{g} , its proof involves the Poisson Lie group G integrating \mathfrak{g} . The paper is organized as follows. In Section 2 for each $g \in G$ we consider \mathfrak{h}^g , the left translation to the identity of $T_g\mathcal{O}$, where \mathcal{O} denotes the symplectic leaf through g . If \mathfrak{h}^g is a Lie subalgebra of \mathfrak{g} then it is automatically a coisotropic subalgebra. In Section 3 we restrict our attention to Lie bialgebras arising from r -matrices and elements g of the form $\exp(X)$, proving the theorem stated above. Section 4 is devoted to explicit examples in which \mathfrak{g} is a semi-simple Lie algebra. In the Appendix we present the geometric motivation that lead to considering the subspaces \mathfrak{h}^g , namely pre-Poisson maps.

Acknowledgments: I learnt the simple proof of Prop. 2.3 from Jiang-Hua Lu. The connection to the work of Evens and Lu established in Remark 4.1 was suggested by the referee. I thank Camille Laurent and Jiang-Hua Lu for helpful conversations. I am indebted to Alberto Cattaneo and to Francesco Bonechi for remarks that improved the final version of this note. Thanks to Philippe Monnier for a visit to Toulouse in October 2008 that helped complete this work.

2. COISOTROPIC SUBALGEBRAS

We recall some notions from the theory of Poisson Lie groups; we refer to the expositions [17, 15, 16] for more details.

Recall that a *Poisson manifold* is a manifold P endowed with a bivector field $\Lambda \in \Gamma(\wedge^2 TP)$ satisfying $[\Lambda, \Lambda] = 0$, where $[\bullet, \bullet]$ denotes the Schouten bracket on multivector fields. We denote by $\Lambda^\sharp: T^*P \rightarrow TP$ the map given by contraction with Λ .

Definition 2.1. A *Poisson Lie group* is a Lie group G equipped with a Poisson bivector Λ such that the multiplication map $m: G \times G \rightarrow G$ is a Poisson map, or equivalently such that

$$(1) \quad \Lambda(gh) = (L_g)_*\Lambda(h) + (R_h)_*\Lambda(g) \text{ for all } g, h \in G.$$

To every element g of the Poisson Lie group G we associate a *subspace* of its Lie algebra \mathfrak{g} as follows:

$$(2) \quad \mathfrak{h}^g := (\eta^g)^\sharp \mathfrak{g}^*,$$

where we use the short-hand notation

$$(3) \quad \eta^g := (L_g)_*\Lambda(g^{-1}) \in \wedge^2 \mathfrak{g}.$$

The subspace \mathfrak{h}^g is the left-translation to the identity of $T_{g^{-1}}\mathcal{O}$, where \mathcal{O} denotes the symplectic leaf of (G, Λ) through g^{-1} ; in particular it is always even dimensional. Notice that $(\eta^g)^\sharp: \mathfrak{g}^* \rightarrow \mathfrak{g}$ satisfies the identity

$$(L_g)_* \circ (\Lambda(g^{-1}))^\sharp = (\eta^g)^\sharp \circ (L_{g^{-1}})^*.$$

Definition 2.2 ([17, Sec. 3.1]). Let \mathfrak{g} be a Lie bialgebra. A Lie subalgebra \mathfrak{h} of \mathfrak{g} is called *coisotropic*¹ if its annihilator \mathfrak{h}° is a Lie subalgebra of \mathfrak{g}^* .

Proposition 2.3. *Let G be a Poisson Lie group and $g \in G$. If $\mathfrak{h}^g \subset \mathfrak{g}$ is a Lie subalgebra then it is automatically a coisotropic subalgebra.*

Proof. Recall that, for every Poisson manifold (P, Λ) , there is a Lie bracket² on the space of 1-forms, inducing a Lie algebra structure on $(T_p \mathcal{O})^\circ$ for each $p \in P$ (here \mathcal{O} denotes the symplectic leaf through p). It is known that the space of left-invariant 1-forms on the Poisson Lie group G is closed with respect to this bracket, and that evaluation at $e \in G$ is a Lie algebra isomorphism onto the Lie algebra \mathfrak{g}^* [17, Sect. 2.5]. In particular $(L_{g^{-1}})^*: (T_{g^{-1}} \mathcal{O})^\circ \rightarrow \mathfrak{g}^*$ is a Lie algebra homomorphism, with image $(\mathfrak{h}^g)^\circ$. Hence $(\mathfrak{h}^g)^\circ$ is a Lie subalgebra of \mathfrak{g}^* . \square

It would be interesting to study the set $\{g \in G : \mathfrak{h}^g \text{ is a Lie subalgebra}\}$. It is closed under inversion but is not a subgroup of G (see Remark 3.7).

Remark 2.4. We are indebted to Jiang Hua Lu for pointing out the above simple proof of Prop. 2.3. In Appendix A we present another proof, based on properties of the left translation L_g .

3. POISSON LIE GROUPS ARISING FROM r -MATRICES

Let (G, Λ) be a Poisson Lie group. In this section we determine elements $g \in G$ for which the subspace $\mathfrak{h}^g \subset \mathfrak{g}$ of eq. (2) is a Lie subalgebra, for Prop. 2.3 tells us that then it is a coisotropic subalgebra.

Lemma 3.1. *If $[\eta^g, \eta^g] = 0 \in \wedge^3 \mathfrak{g}$ then \mathfrak{h}^g is a Lie subalgebra of \mathfrak{g} .*

Proof. $[\eta^g, \eta^g] = 0$ iff $\overrightarrow{\eta^g}$, the right-invariant bivector on G whose value at the identity is η^g , is a Poisson bivector. In that case the symplectic distribution $(\overrightarrow{\eta^g})^\# T^*G = \overrightarrow{\mathfrak{h}^g}$ is involutive, and this is equivalent to \mathfrak{h}^g being a Lie subalgebra of \mathfrak{g} . \square

Definition 3.2. Let \mathfrak{g} be a Lie algebra. An r -matrix is an element $\pi \in \wedge^2 \mathfrak{g}$ such that $[\pi, \pi]$ is ad -invariant.

It is known [7] that if π is an r -matrix for the Lie algebra \mathfrak{g} then $\Lambda := \overleftarrow{\pi} - \overrightarrow{\pi}$ makes G , any Lie group integrating \mathfrak{g} , into a Poisson Lie group. From now on we restrict ourselves to such Poisson Lie groups. Notice that from definition (3) we get

$$(4) \quad \eta^g = \pi - Ad_g \pi.$$

Now we are able to state the main result of this paper.

Theorem 3.3. *Let G be a Poisson Lie group corresponding to an r -matrix π , $X \in \mathfrak{g}$, $g := \exp(X)$. Assume that*

$$(5) \quad [X, [X, \pi]] = \lambda[X, \pi] \text{ for some } \lambda \in \mathbb{R}.$$

¹A Lie subalgebra \mathfrak{h} is coisotropic iff the connected subgroup H integrating it is a coisotropic subgroup of (G, Λ) (see for instance [5]).

Another equivalent characterization of the fact that \mathfrak{h} is a coisotropic Lie subalgebra is the following: \mathfrak{h} is a coisotropic submanifold of \mathfrak{g} , endowed with the linear Poisson structure induced by the Lie algebra \mathfrak{g}^* , and \mathfrak{h}° is a coisotropic submanifold of the linear Poisson manifold \mathfrak{g}^* .

²Indeed, T^*P with this bracket and the bundle map $\Lambda^\sharp: T^*P \rightarrow TP$ forms a Lie algebroid [2].

Then \mathfrak{h}^g is a coisotropic subalgebra of \mathfrak{g} . Further

$$(6) \quad \mathfrak{h}^g = [X, \pi]^\# \mathfrak{g}^*.$$

Proof. Notice that

$$Ad_{exp(X)}\pi = e^{ad_X}\pi = \pi + [X, \pi] + \frac{1}{2}[X, [X, \pi]] + \frac{1}{3!}[X, [X, [X, \pi]]] + \cdots = \pi + \frac{e^\lambda - 1}{\lambda}[X, \pi].$$

Therefore

$$\eta^g = \pi - Ad_g\pi = \pi - (\pi + \frac{e^\lambda - 1}{\lambda}[X, \pi]) = -\frac{e^\lambda - 1}{\lambda}[X, \pi].$$

Now we use twice the fact that $[\pi, [X, \pi]] = \frac{1}{2}[X, [\pi, \pi]] = 0$ (by the graded Jacobi identity) to show that

$$[[X, \pi], [X, \pi]] = [X, [\pi, [X, \pi]]] - [\pi, [X, [X, \pi]]] = 0 - \lambda \cdot 0 = 0.$$

This means that $[\eta^g, \eta^g] = 0$, and by Lemma 3.1 and Prop. 2.3 \mathfrak{h}^g is a coisotropic subalgebra. The last part of the theorem follows since the function $\frac{e^\lambda - 1}{\lambda}$ never vanishes. \square

Remark 3.4. If $X \in \mathfrak{g}$ satisfies condition (5) then $\Lambda = \overleftarrow{\pi} - \overrightarrow{\pi}$ and $\overrightarrow{\eta^g}$ (or $\overleftarrow{\eta^g}$) are commuting Poisson structures on G . This follows at once from the computations of the proof of Thm. 3.3, noticing that η^g is a multiple of $[X, \pi]$. Here at usual $g := exp(X)$.

We now display two very simple examples.

Example 3.5. Let $\mathfrak{g} = \mathfrak{su}(2)$, so that for a suitable basis we have $[e_1, e_2] = e_3, [e_2, e_3] = 1, [e_3, e_1] = e_2$, and take the r -matrix $\pi = 2e_2 \wedge e_3$ as in [17, Ex. 2.10]. Then the only elements of $\mathfrak{su}(2)$ that satisfy eq. (5) are the multiples X of e_1 , and applying (6) we see that they all give $\mathfrak{h}^{exp(X)} = \{0\}$.

Example 3.6. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, with basis

$$e_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = -e_2$, and $\pi = 2e_2 \wedge e_3$ is an r -matrix [17, Ex. 2.9]. The vectors X of $\mathfrak{sl}(2, \mathbb{R})$ that satisfy eq. (5) are exactly those of the form $\alpha e_1 + \beta(e_2 + e_3)$ (the upper triangular matrices) and $\alpha e_1 + \beta(e_2 - e_3)$ (the lower triangular matrices). Applying Thm. 3.3 we obtain coisotropic subalgebras $span\{e_1, e_2 - e_3\}$, $span\{e_1, e_2 + e_3\}$ and $\{0\}$.

Using (3) one can compute directly all the elements $g \in G = SL(2, \mathbb{R})$ for which $[\eta^g, \eta^g] = 0$: they are those of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}$. By Lemma 3.1 and Prop. 2.3 these group elements g give rise to a coisotropic subalgebra of \mathfrak{g} . The first class of elements g with $b \neq 0$ all give rise to $span\{e_1, e_2 - e_3\}$, the second class of elements g with $c \neq 0$ all give rise to $span\{e_1, e_2 + e_3\}$, and the diagonal matrices give rise to the trivial subalgebra $\{0\}$, i.e. we obtain exactly the same coisotropic subalgebras as above.

Remark 3.7. We show that $\{g \in G : \mathfrak{h}^g \text{ is a Lie subalgebra}\}$ is closed under the inversion map but not under multiplication. Indeed notice that $\eta^{g^{-1}} = -Ad_{g^{-1}}\eta^g$ by (1), so $\mathfrak{h}^{g^{-1}} = Ad_{g^{-1}}\mathfrak{h}^g$, and since $Ad_{g^{-1}}$ is a Lie algebra isomorphism the first statement follows.

To show the second statement consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ as in Example 3.6. The elements $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ of $G = SL(2, \mathbb{R})$ have the property that \mathfrak{h}^g and \mathfrak{h}^h are Lie subalgebras, by Example 3.6. However $\eta^{gh} = \pi - Ad_{gh}\pi = 2(e_1 \wedge e_2 + 2e_2 \wedge e_3 - e_1 \wedge e_3)$, implying that \mathfrak{h}^{gh} is not a Lie subalgebra of \mathfrak{g} .

4. EXAMPLES: SEMI-SIMPLE COMPLEX LIE ALGEBRAS

In this section we consider the standard Lie bialgebra structure on a semi-simple *complex* Lie algebra, and out of its roots, using Thm. 3.3 we construct families of coisotropic subalgebras. We write down explicitly³ the resulting families for the classical simple Lie algebras $\mathfrak{sl}(n+1, \mathbb{C})$, $\mathfrak{so}(2n+1, \mathbb{C})$, $\mathfrak{sp}(2n, \mathbb{C})$, $\mathfrak{so}(2n, \mathbb{C})$ and for their split real forms $\mathfrak{sl}(n+1, \mathbb{R})$, $\mathfrak{so}(n+1, n)$, $\mathfrak{sp}(2n, \mathbb{R})$, $\mathfrak{so}(n, n)$. We refer to [1, Ch. 2.6], to [12] and to [13] for background material about semi-simple complex Lie algebras and their real forms.

Let \mathfrak{g} be a semi-simple Lie algebra over \mathbb{C} , and fix a Cartan subalgebra \mathfrak{h} . There is a decomposition $\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in R} \mathfrak{g}^\alpha$ where \mathfrak{g}^α denotes the one dimensional eigenspace for the adjoint action of \mathfrak{h} associated to the “eigenvalue” $\alpha \in \mathfrak{h}^*$. The set $R \subset \mathfrak{h}^*$ is called root system; make a choice R_+ of positive roots. For each $\alpha \in R_+$ choose non-zero $e_\alpha \in \mathfrak{g}^\alpha$ and $f_\alpha \in \mathfrak{g}^{-\alpha}$.

Then an r -matrix is given by

$$(7) \quad \pi := \sum_{\alpha \in R_+} \lambda_\alpha e_\alpha \wedge f_\alpha$$

where $\lambda_\alpha := \frac{1}{B(e_\alpha, f_\alpha)}$ [16, Ex. 2.10]. Notice that, since the subspaces \mathfrak{g}^α are one dimensional and the Killing form B is \mathbb{C} -bilinear, the above r -matrix depends only on the choice of Cartan subalgebra.

Remark 4.1. As above let \mathfrak{g} be a semi-simple complex Lie algebra. Evens and Lu [10][11, Sec. 2.1] consider the direct sum Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ endowed with the pairing⁴ $\langle x_1 + y_1, x_2 + y_2 \rangle = \frac{1}{2}B(x_1, y_1) - \frac{1}{2}B(x_2, y_2)$ where B is the Killing form of \mathfrak{g} . They study the variety $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ of lagrangian subalgebras, and endow it with interesting Poisson structures.

Since $(\mathfrak{g}, [\pi, \bullet])$ is a Lie bialgebra, $\mathfrak{g} \oplus \mathfrak{g}^*$ admits a Lie algebra structure known as Drinfeld double, for which the natural pairing is ad -invariant [17, Sec. 2.3]. If $\mathfrak{k} \subset \mathfrak{g}$ is a coisotropic subalgebra, then $\mathfrak{k} \oplus \mathfrak{k}^\circ \subset \mathfrak{g} \oplus \mathfrak{g}^*$ is a lagrangian subalgebra.

There is an isomorphism of Lie algebras

$$(8) \quad \mathfrak{g} \oplus \mathfrak{g}^* \cong \mathfrak{g} \oplus \mathfrak{g}$$

preserving the pairings. As a consequence, coisotropic subalgebras of \mathfrak{g} give rise to points of $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$, which as seen above is an interesting and well-studied variety.

Eq. (8) follows from [18, Prop. 1.5] (see also [21, Prop. 2.1]). We reproduce the proof for completeness. Recall that a Manin triple consists of a Lie algebra with an ad -invariant non-degenerate symmetric pairing and a decomposition into two Lagrangian subalgebras. There is a bijection between Manin triples and Lie bialgebras [14, Thm. 2.3.2]. $\mathfrak{g} \oplus \mathfrak{g}$, together with the diagonal \mathfrak{g}_Δ and

$$(9) \quad \{(h + v, -h + w) : h \in \mathfrak{h}, v \in \oplus_{\alpha \in R_+} \mathfrak{g}^\alpha, w \in \oplus_{\alpha \in R_+} \mathfrak{g}^{-\alpha}\},$$

forms a Manin triple. The corresponding Lie bialgebra consists of the Lie algebra \mathfrak{g} with the derivation of $\wedge^\bullet \mathfrak{g}$ obtain dualizing the Lie bracket on (9). A computation shows that this derivation is exactly $[\pi, \bullet]$. Hence the Drinfeld double $\mathfrak{g} \oplus \mathfrak{g}^*$ of the Lie bialgebra $(\mathfrak{g}, [\pi, \bullet])$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$ by a pairing-preserving map, showing (8).

Lemma 4.2. *Let $X \in \mathfrak{g}$ and assume that for all $\alpha \in R_+$*

$$1) \quad [X, [X, e_\alpha]] \wedge f_\alpha = 0$$

³One reason for doing this is that we were not able to find any explicit families of examples of coisotropic subalgebras in the literature.

⁴They actually consider any non-zero multiple of the Killing form, not just $\frac{1}{2}$.

- 2) $[X, [X, f_\alpha]] \wedge e_\alpha = 0$
- 3) $[X, e_\alpha] \wedge [X, f_\alpha] = 0.$

Then X satisfies condition (5) (with $\lambda = 0$).

Proof. We compute

$$[X, \pi] = \sum_{\alpha \in R_+} \lambda_\alpha ([X, e_\alpha] \wedge f_\alpha + e_\alpha \wedge [X, f_\alpha]),$$

so

$$[X, [X, \pi]] = \sum_{\alpha \in R_+} \lambda_\alpha ([X, [X, e_\alpha]] \wedge f_\alpha + 2[X, e_\alpha] \wedge [X, f_\alpha] + e_\alpha \wedge [X[X, f_\alpha]]),$$

each term of which vanishes by our assumptions. \square

Proposition 4.3. *Let $\beta \in R_+$ satisfy this condition:*

- (10) *For all $\alpha \in R$: $(\alpha + \mathbb{Z}\beta) \cap R$ does not contain a string of 3 consecutive elements.*

Then e_β and f_β satisfy condition (5).

Proof. We check that $X = e_\beta$ satisfies the assumptions of Lemma 4.2; the proof for f_β is similar. Let $\alpha \in R$.

Suppose that $[e_\beta, [e_\beta, e_\alpha]] \neq 0$. Then $\alpha, \alpha + \beta$ and $\alpha + 2\beta$ form a string of 3 consecutive elements in $(\alpha + \mathbb{Z}\beta) \cap (R \cup \{0\})$. Since the intersection of R with any line through the origin is either empty or of the form $\{\alpha, -\alpha\}$ [1, Prop. 2.20] it follows that $\beta = -\alpha$. So $[e_\beta, [e_\beta, e_\alpha]]$ is a multiple of f_α , and assumption 1) of Lemma 4.2 is satisfied.

Similarly, if $[e_\beta, [e_\beta, f_\alpha]] \neq 0$, then $-\alpha, -\alpha + \beta$ and $-\alpha + 2\beta$ form a string of 3 consecutive elements in $(\alpha + \mathbb{Z}\beta) \cap (R \cup \{0\})$, so we must have $\beta = \alpha$. So $[e_\beta, [e_\beta, f_\alpha]]$ is a multiple of e_α , and assumption 2) of Lemma 4.2 is satisfied.

At most one of $\alpha + \beta$ or $\alpha - \beta$ lie in R : if they both did then $\{\alpha - \beta, \alpha, \alpha + \beta\}$ would be a string of 3 consecutive elements in $(\alpha + \mathbb{Z}\beta) \cap R$, contradicting our assumption. If $\alpha - \beta \notin R$ then either $\alpha - \beta = 0$, in which case $[e_\alpha, e_\beta] = 0$, or $[e_\alpha, f_\beta] \in \mathfrak{g}^{\alpha - \beta} = \{0\}$. A similar reasoning holds for $\alpha + \beta$, so we conclude that assumption 3) of Lemma 4.2 holds. \square

Corollary 4.4. *Assume the notation above and assume that $\beta \in R_+$ satisfy condition (10).*

Let $\mathfrak{g}_\mathbb{R}$ denote \mathfrak{g} viewed as a real Lie algebra. Then $[e_\beta, \pi]^\sharp \mathfrak{g}_\mathbb{R}^$ and $[f_\beta, \pi]^\sharp \mathfrak{g}_\mathbb{R}^*$*

- *are coisotropic subalgebras of $\mathfrak{g}_\mathbb{R}$*
- *their complexifications are coisotropic subalgebras of the complex Lie bialgebra \mathfrak{g} .*

Proof. The first statement follows from Prop. 4.3 and applying Thm. 3.3 to $\mathfrak{g}_\mathbb{R}$.

Now choose $\tilde{e}_\alpha \in \mathfrak{g}^\alpha$ and $\tilde{f}_\alpha \in \mathfrak{g}^{-\alpha}$ to be part of a Chevalley basis [1, Ch. 2.6] of \mathfrak{g} , so that

$$\mathfrak{g}_0 := \{h \in \mathfrak{h} : \alpha(h) \in \mathbb{R} \text{ for all } \alpha \in R_+\} \oplus_{\alpha \in R_+} \text{span}_\mathbb{R} \{\tilde{e}_\alpha, \tilde{f}_\alpha\}$$

is a Lie subalgebra of $\mathfrak{g}_\mathbb{R}$, namely a split real form of \mathfrak{g} [13, p. 296]. Since $\pi \in \wedge^2 \mathfrak{g}_0$ and $\tilde{e}_\beta \in \mathfrak{g}_0$, applying Thm. 3.3 to \mathfrak{g}_0 we deduce that $[\tilde{e}_\beta, \pi]^\sharp \mathfrak{g}_0^*$ is a coisotropic subalgebra of \mathfrak{g}_0 . The complexification of $[\tilde{e}_\beta, \pi]^\sharp \mathfrak{g}_0^* = [\tilde{e}_\beta, \pi]^\sharp \mathfrak{g}_\mathbb{R}^*$ coincides with the complexification of $[e_\beta, \pi]^\sharp \mathfrak{g}_\mathbb{R}^*$, hence the second statement follows. \square

Our main references for the computation of the examples below are [12, part III] and [20]. Two remarks about the derivation of the examples are in order.

Remark 4.5. 1) We use the fact that the Killing form $B(A_1, A_2)$ is a non-zero real multiple of $\text{Tr}(A_1 A_2)$ [12, Ex. 14.36]. Since the elements e_α and f_α we choose are always *real* matrices, the bivector π is also real, and the coisotropic subalgebras of $\mathfrak{g}_\mathbb{R}$ we obtain are also coisotropic subalgebras of $\mathfrak{g} \cap \text{Mat}(n, \mathbb{R})$, which agrees with the split real form of \mathfrak{g} .

2) The coisotropic subspace associated to f_β will be obtained just applying the transposition map to the one associated to e_β . Indeed in all the examples below the transposition map \bullet^T is an anti-homomorphism of \mathfrak{g} which switches the e_α 's and the f_α 's, so it maps π to $-\pi$ and $[e_\beta, \pi]$ to $[f_\beta, \pi]$.

Example 4.6 (A_n). Let $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ with Cartan subalgebra \mathfrak{h} given by the diagonal matrices, so that as roots we obtain $R = \{L_i - L_j\}_{(i \neq j)} \subset \mathbb{R}^{n+1}$, where L_1, \dots, L_{n+1} denotes the standard basis of \mathbb{R}^{n+1} . It is easy to check that all roots satisfy assumption (10).

For a root $\alpha = L_i - L_j$ with $i < j$ we choose $e_\alpha := E_{ij} \in \mathfrak{g}^{L_i - L_j}$ and $f_\alpha := E_{ji} \in \mathfrak{g}^{-L_i + L_j}$, where E_{ij} denotes the matrix with 1 in the (i, j) -entry and zeros elsewhere. We have $\pi \sim \sum_{i < j} E_{ij} \wedge E_{ji}$, where “ \sim ” means “is a non-zero real multiple of”. Fix a root $\beta = L_i - L_j$ with $i < j$. A computation shows that

$$[E_{ij}, \pi] \sim \left(\sum_{i < k < j} + \sum_{i \leq k < j} \right) E_{ik} \wedge E_{kj} = 2 \sum_{i < k < j} E_{ik} \wedge E_{kj} - E_{ij} \wedge (H_i - H_j),$$

where $H_i := E_{ii}$, so for all $i < j$ we obtain a coisotropic subalgebra of \mathfrak{g} spanned by

$$\boxed{E_{ij}, \quad H_i - H_j, \quad \{E_{kj}\}_{i < k < j} \text{ and } \{E_{ik}\}_{i < k < j}}.$$

For instance, letting $n = 2$ and taking $e_\beta = E_{13}$ leads to the coisotropic subalgebra

$$\left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & -a \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

The coisotropic subalgebra we obtain from $f_\beta = E_{ji}$ ($i < j$) is spanned by

$$\boxed{E_{ji}, \quad H_i - H_j, \quad \{E_{ki}\}_{i < k < j} \text{ and } \{E_{jk}\}_{i < k < j}}.$$

All of the above are also coisotropic subalgebras of the split real form $\mathfrak{sl}(n+1, \mathbb{R})$.

Example 4.7 (B_n). Let $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$, with Cartan subalgebra given by the diagonal matrices. Then $R = \{\pm L_i \pm L_j\}_{(i < j)} \cup \{\pm L_i\} \subset \mathbb{R}^n$. The roots that satisfy assumption (10) are exactly those of the form $\pm L_i \pm L_j$ ($i < j$).

The root space of a root $L_i - L_j$ (with $i \neq j$) is spanned by $X_{ij} = E_{i,j} - E_{n+j,n+i}$. The root space of a root $L_i + L_j$ is spanned by $Y_{ij} = E_{i,j+n} - E_{j,n+i}$, the one of $-L_i - L_j$ is spanned by $Z_{ij} = E_{n+i,j} - E_{n+j,i}$. Finally, the root space of L_i is spanned by $U_i = E_{i,2n+1} - E_{2n+1,n+i}$ and the one of $-L_i$ is spanned by $V_i = E_{n+i,2n+1} - E_{2n+1,i}$. As earlier, E_{ij} denotes the matrix with 1 in the (i, j) -entry and zeros elsewhere. The r -matrix of eq. (7) satisfies

$$\pi \sim \frac{1}{2} \left(\sum_{i < j} X_{ij} \wedge X_{ji} - \sum_{i < j} Y_{ij} \wedge Z_{ij} - \sum_i U_i \wedge V_i \right).$$

Given a root $\beta = L_i - L_j$ (with $i < j$), a lengthy but straightforward computation shows

$$[X_{ij}, \pi] \sim -2 \sum_{i < k < j} (X_{ik} \wedge X_{kj}) + X_{ij} \wedge (H_i - H_j).$$

So for all $i < j$ we obtain a coisotropic subalgebra spanned by

$$\boxed{\{X_{ik}, X_{kj}\}_{(i < k < j)}, \quad X_{ij}, \quad H_i - H_j}$$

where $H_i := E_{i,i} - E_{n+i,n+i} \in \mathfrak{h}$. The negative root vector $f_\beta = X_{ji}$ delivers the coisotropic subalgebra spanned by

$$\boxed{\{X_{ki}, X_{jk}\}_{(i < k < j)}, \quad X_{ji}, \quad H_i - H_j}.$$

If instead we pick a root $\beta = L_i + L_j$ (with $i < j$) we obtain

$$[Y_{ij}, \pi] = -2 \sum_{i < k \neq j} (X_{ik} \wedge Y_{kj}) + 2 \sum_{j < k} (X_{jk} \wedge Y_{ki}) + Y_{ij} \wedge (H_i - H_j) + 2U_i \wedge U_j,$$

giving rise to a coisotropic subalgebra spanned by

$$\boxed{\{X_{ik}, Y_{kj}\}_{(i < k \neq j)}, \quad \{X_{jk}, Y_{ki}\}_{(j < k)}, \quad Y_{ij}, \quad H_i - H_j, \quad U_i, \quad U_j}.$$

The root $-(L_i + L_j)$ (with $i < j$) delivers the coisotropic subalgebra spanned by

$$\boxed{\{X_{ki}, Z_{kj}\}_{(i < k \neq j)}, \quad \{X_{kj}, Z_{ki}\}_{(j < k)}, \quad Z_{ij}, \quad H_i - H_j, \quad V_i, \quad V_j}.$$

All of the above are also coisotropic subalgebras of the split real form $\mathfrak{so}(n+1, n)$.

Example 4.8 (C_n). Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$. Then, choosing the diagonal matrices as Cartan subalgebra, $R = \{\pm L_i \pm L_j\} \subset \mathbb{R}^n$. The only roots that satisfy assumption (10) are those of the form $\pm 2L_i$.

For $i \neq j$ the root space of a root $L_i - L_j$ is spanned by $X_{ij} = E_{i,j} - E_{n+j,n+i}$, as in Ex. 4.7; the root space of a root $L_i + L_j$ is spanned by $Y_{ij} = E_{i,n+j} + E_{j,n+i}$, the one of $-L_i - L_j$ is spanned by $Z_{ij} = E_{n+i,j} + E_{n+j,i}$. Finally, the root space of $2L_i$ is spanned by $U_i = E_{i,n+i}$ and the one of $-2L_i$ is spanned by $V_i = E_{n+i,i}$. We obtain the r -matrix

$$\pi \sim \frac{1}{2} \sum_{i < j} X_{ij} \wedge X_{ji} + \frac{1}{2} \sum_{i < j} Y_{ij} \wedge Z_{ij} + \sum_i U_i \wedge V_i.$$

Let us consider the root $2L_i$. A computation shows

$$[U_i, \pi] \sim \sum_{i < k} (Y_{ik} \wedge X_{ik}) + U_i \wedge H_i,$$

where $H_i := E_{ii} - E_{n+i,n+i}$, so as coisotropic subspace we obtain the span of

$$\boxed{\{Y_{ik}, X_{ik}\}_{i < k}, \quad U_i, \quad H_i}.$$

For instance, when $n = 2$, taking $e_\beta = U_2 = E_{24}$ and $e_\beta = U_1 = E_{13}$ we obtain the coisotropic subalgebras of $\mathfrak{sp}(4, \mathbb{C})$

$$\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a \end{pmatrix} : a, b \in \mathbb{R} \right\} \text{ and } \left\{ \begin{pmatrix} a & c & b & d \\ 0 & 0 & d & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & -c & 0 \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

For the root $-2L_i$, whose root space is spanned by V_i , as coisotropic subspace we obtain the span of

$$\boxed{\{Z_{ik}, X_{ki}\}_{i < k}, \quad V_i, \quad H_i}.$$

All of the above are also coisotropic subalgebras of the split real form $\mathfrak{sp}(2n, \mathbb{R})$.

Example 4.9 (D_n). Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$. Then $R = \{\pm L_i \pm L_j\}_{\{i < j\}} \subset \mathbb{R}^n$, and the same computation as in Ex. 4.7 shows that all roots satisfy assumption (10). The root spaces of

$L_i - L_j, L_i + L_j$ and $-L_i - L_j$ are spanned by elements X_{ij}, Y_{ij} and Z_{ij} defined by the same formulae as in Ex. 4.7, and the r -matrix of eq. (7) satisfies

$$\pi \sim \frac{1}{2} \left(\sum_{i < j} X_{ij} \wedge X_{ji} - \sum_{i < j} Y_{ij} \wedge Z_{ji} \right)$$

(it consists of the first two summands of the r -matrix for the B_n case).

The same computations as in Ex. 4.7 show that (with $i < j$) from the root $L_i - L_j$ we obtain the coisotropic subalgebras spanned by

$$\boxed{\{X_{ik}, X_{kj}\}_{(i < k < j)}, \quad X_{ij}, \quad H_i - H_j}$$

and

$$\boxed{\{X_{ki}, X_{jk}\}_{(i < k < j)}, \quad X_{ji}, \quad H_i - H_j},$$

whereas from the root $L_i + L_j$ we obtain the coisotropic subalgebras spanned by

$$\boxed{\{X_{ik}, Y_{kj}\}_{(i < k \neq j)}, \quad \{X_{jk}, Y_{ki}\}_{(j < k)}, \quad Y_{ij}, \quad H_i - H_j}$$

and

$$\boxed{\{X_{ki}, Z_{kj}\}_{(i < k \neq j)}, \quad \{X_{kj}, Z_{ki}\}_{(j < k)}, \quad Z_{ij}, \quad H_i - H_j}.$$

(Here $H_i := E_{i,i} - E_{n+i,n+i}$). All of the above are also coisotropic subalgebras of the real form $\mathfrak{so}(n, n)$.

Remark 4.10. In Example 4.6, taking $n = 2$ and $g = \exp(E_{13})$, we showed that $\mathfrak{h}^g = \text{span}_{\mathbb{R}}\{E_{12}, E_{13}, E_{23}, H_1 - H_3\}$ is a coisotropic subalgebra of $\mathfrak{sl}(3, \mathbb{R})$. In particular its annihilator $(\mathfrak{h}^g)^\circ$ is a Lie subalgebra, but it is *not* a Lie ideal. Indeed, taking the basis of $\mathfrak{sl}(3, \mathbb{R})$ given by $\{E_{ij}\}_{(i \neq j)}$, $H_1 - H_2$, $H_1 - H_3$ and considering the dual basis, we have $(H_1 - H_2)^* \in (\mathfrak{h}^g)^\circ$ but $\langle [(E_{12})^*, (H_1 - H_2)^*], E_{12} \rangle \neq 0$.

APPENDIX A. PRE-POISSON MAPS

In this appendix we generalize the notion of Poisson map between Poisson manifolds. A natural example is the left translation L_g on a Poisson Lie group G (Lemma A.7), which gives rise naturally to the subspace $\mathfrak{h}^g \subset T_e G$ considered in Section 2, providing an alternative proof of Prop. 2.3.

Recall that a submanifold C of a Poisson manifold P is called *coisotropic* if $\Lambda^\sharp N^*C \subset TC$, where N^*C (the conormal bundle of C) is defined as the annihilator of TC . Here we need a generalization of the notion of coisotropic submanifold:

Definition A.1. A submanifold C of a Poisson manifold (P, Λ) is called *pre-Poisson* [4] if the rank of $TC + \Lambda^\sharp N^*C$ is constant along C , or equivalently if $pr_{NC} \circ \Lambda^\sharp: N^*C \rightarrow TP|_C \rightarrow NC := TP|_C/TC$ has constant rank.

A map $\phi: (P_1, \Lambda_1) \rightarrow (P_2, \Lambda_2)$ between Poisson manifolds is a *pre-Poisson map* if $\text{graph}(\phi)$ is a pre-Poisson submanifold of the product $P_1 \times \bar{P}_2$, where \bar{P}_2 denotes the Poisson manifold $(P_2, -\Lambda_2)$.

A map between Poisson manifolds is a Poisson map iff its graph is coisotropic, hence we see that pre-Poisson maps generalize the notion of Poisson map. We make more explicit what it means to be a pre-Poisson map.

Lemma A.2. A map $\phi: (P_1, \Lambda_1) \rightarrow (P_2, \Lambda_2)$ is pre-Poisson iff for all $x \in P_1$ the rank of

$$E(x) = \{(\Lambda_2 - \phi_*\Lambda_1)^\sharp \xi : \xi \in T_{\phi(x)}^* P_2\} \subset T_{\phi(x)} P_2$$

is constant. Here $\phi_*: T_x P_1 \rightarrow T_{\phi(x)} P_2$.

Proof. Let $\Gamma := \text{graph}(\phi) \subset P_1 \times \bar{P}_2$ and $x \in P_1$. We have

$$\begin{aligned} T_{(x, \phi(x))} \Gamma + (\Lambda_1 - \Lambda_2)^\sharp N_{(x, \phi(x))}^* \Gamma &= \{(X, \phi_* X) : X \in T_x P_1\} + \{(\Lambda_1^\sharp \phi^* \xi, \Lambda_2^\sharp \xi) : \xi \in T_{\phi(x)}^* P_2\} \\ &= \{(X, \phi_* X) : X \in T_x P_1\} + \{(0, \Lambda_2^\sharp \xi - \phi_*(\Lambda_1^\sharp \phi^* \xi)) : \xi \in T_{\phi(x)}^* P_2\} \\ &= \{(X, \phi_* X) : X \in T_x P_1\} + \{0\} \times E(x). \end{aligned}$$

A complement of this subspace in $T_{(x, \phi(x))}(P_1 \times P_2)$ is $(0, R(x))$, where $R(x)$ is a complement to $E(x)$ in $T_{\phi(x)} P_2$. Hence Γ is a pre-Poisson submanifold iff $R(x)$, or equivalently $E(x)$, has constant rank as x varies over all points of P_1 . \square

Remark A.3. 1) The composition of pre-Poisson maps is *not* pre-Poisson. Let $P_1 = (\mathbb{R}^2, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$, $P_2 = (\mathbb{R}^2, 0)$ and $P_3 = (\mathbb{R}^2, (1 + x^2 + y^2) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$. The identity maps $\text{id}: P_1 \rightarrow P_2$ and $\text{id}: P_2 \rightarrow P_3$ are pre-Poisson maps (this is seen easily using Lemma A.2), however the composition is not.

2) Let P_1, P_2 be Poisson manifolds and $\phi: P_1 \rightarrow P_2$ be a submersive *Poisson* map. If $C \subset P_2$ is a pre-Poisson submanifold (for example a point), then $f^{-1}(C)$ is a pre-Poisson submanifold of P_1 [3]. When ϕ is just a submersive *pre-Poisson* map this statement is not longer true: the projection $\phi: (\mathbb{R}^3, -z^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}) \rightarrow (\mathbb{R}^2, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$ onto the first two components is a pre-Poisson map, but $\phi^{-1}(0) = \{(0, 0, z) : z \in \mathbb{R}\}$ is not a pre-Poisson submanifold.

From now on we consider only the case when the map ϕ of Lemma A.2 is a *diffeomorphism*. Then $D_y := E(\phi^{-1}(y))$ defines a singular distribution on P_2 which measures how ϕ fails to be a Poisson map.

Definition A.4. Given a diffeomorphism $\phi: (P_1, \Lambda_1) \rightarrow (P_2, \Lambda_2)$ between Poisson manifolds, the *deficit distribution* associated to ϕ is the singular distribution on P_2 given by

$$D = \{(\Lambda_2 - \phi_* \Lambda_1)^\sharp \xi : \xi \in T^* P_2\}.$$

The deficit distribution D singles out an interesting subalgebra of $C^\infty(P_2)$:

Lemma A.5. Let $\phi: (P_1, \Lambda_1) \rightarrow (P_2, \Lambda_2)$ be a diffeomorphism. Then the set of D -invariant functions $\{f : d_y f|_{D_y} = 0 \text{ for all } y \in P_2\}$ coincides with

$$(11) \quad \{f : \phi^* \{f, g\} = \{\phi^* f, \phi^* g\} \text{ for all } g \in C^\infty(P_2)\},$$

and is a Poisson subalgebra of $C^\infty(P_2)$.

Proof. Expressing D in terms of hamiltonian vector fields we have $D = \{X_g^{P_2} - \phi_*(X_{\phi^* g}^{P_1}) : g \in C^\infty(P_2)\}$. The claimed equality follows from

$$d_y f(X_g^{P_2} - \phi_*(X_{\phi^* g}^{P_1})) = \{f, g\}_y - d_{\phi^{-1}(y)}(\phi^* f) X_{\phi^* g}^{P_1} = (\phi^* \{f, g\} - \{\phi^* f, \phi^* g\})_{\phi^{-1}(y)}$$

for all $y \in P_2$.

To show that (11) is a Poisson subalgebra we compute for D -invariant functions f and \tilde{f} on P_2 and for $g \in C^\infty(P_2)$ that

$$\phi^* \{\{f, g\}, \tilde{f}\} = \{\phi^* \{f, g\}, \phi^* \tilde{f}\} = \{\{\phi^* f, \phi^* g\}, \phi^* \tilde{f}\}.$$

Hence using twice the Jacobi identity we obtain

$$\begin{aligned} \phi^* \{\{f, \tilde{f}\}, g\} &= \phi^* \{\{f, g\}, \tilde{f}\} + \phi^* \{f, \{\tilde{f}, g\}\} \\ &= \{\{\phi^* f, \phi^* g\}, \phi^* \tilde{f}\} + \{\phi^* f, \{\phi^* \tilde{f}, \phi^* g\}\} = \{\{\phi^* f, \phi^* \tilde{f}\}, \phi^* g\} = \{\phi^* \{f, \tilde{f}\}, \phi^* g\}. \end{aligned}$$

□

Summarizing the above results we have

Corollary A.6. *A diffeomorphism $\phi: (P_1, \Lambda_1) \rightarrow (P_2, \Lambda_2)$ is a pre-Poisson map iff $\Lambda_2 - \phi_*\Lambda_1$ is a constant rank bivector on P_2 , i.e. iff D is a smooth constant rank distribution on P_2 . If D is integrable and the leaf space P_2/D is smooth, then P_2/D has a Poisson structure induced by the projection map $\pi: P_2 \rightarrow P_2/D$. In this case the composition $\pi \circ \phi: P_1 \rightarrow P_2/D$ is a Poisson map.*

Proof. ϕ is a pre-Poisson map by Lemma A.2. By the second part of Lemma A.5 the D -invariant functions on P_2 form a Poisson subalgebra of $C^\infty(P_2)$, so P_2/D has an induced Poisson structure. By the first part of Lemma A.5 in particular $\phi^*\{f, \tilde{f}\} = \{\phi^*f, \phi^*\tilde{f}\}$ for all D -invariant functions f, \tilde{f} on P_2 , so $\pi \circ \phi$ is a Poisson map. □

Now let G be a Poisson Lie group and $g \in G$. The subspace \mathfrak{h}^g defined in Section 2 generates the deficit distribution of the left translation $L_g: G \rightarrow G$.

Lemma A.7. *a) $L_g: G \rightarrow G$ is a pre-Poisson map.*

b) Its deficit distribution is $\overrightarrow{\mathfrak{h}}^g$, the right-invariant distribution obtained translating $\mathfrak{h}^g \subset T_e G$.

Proof. a) By Cor. A.6 we have to show that $\Lambda - (L_g)_*\Lambda$ is a constant rank bivector on G . This bivector field at the point $k \in G$ is

$$(12) \quad \Lambda(k) - (L_g)_*[\Lambda(g^{-1}k)] = -(L_g)_*(R_k)_*\Lambda(g^{-1}) = -(R_k)_*\eta^g,$$

where we have used (1) applied to $\Lambda(g^{-1}k)$ in the first equality. In other words $\Lambda - (L_g)_*\Lambda = -\overrightarrow{\eta}^g$, which obviously has constant rank.

b) Using a) we see that the deficit distribution is $[\Lambda - (L_g)_*\Lambda]^\# T^*G = [\overrightarrow{\eta}^g]^\# T^*G = \overrightarrow{\mathfrak{h}}^g$. □

The observations above allow for an alternative, perhaps more geometric, proof of Prop. 2.3.

Alternative proof of Prop. 2.3. For any $f_1, f_2 \in C^\infty(G)$ and $X \in \mathfrak{g}$ we have [17, Ch. 2.3]

$$(13) \quad \langle [d_e f_1, d_e f_2], X \rangle = X\{f_1, f_2\}.$$

Any element of $(\mathfrak{h}^g)^\circ$ can be realized as $d_e f$ where f is a function on G which is invariant along the integrable distribution obtained right-translating \mathfrak{h}^g . This distribution coincides with the deficit distribution of $L_g: G \rightarrow G$ by Lemma A.7 b). Hence, if f_1 and f_2 are invariant functions, by Lemma A.5 $\{f_1, f_2\}$ is also invariant. Therefore the right hand side of (13) vanishes for all $X \in \mathfrak{h}^g$, from which we deduce that $[d_e f_1, d_e f_2] \in (\mathfrak{h}^g)^\circ$. □

We conclude with two remarks on Poisson actions.

Remark A.8. The considerations of Lemma A.7 can be extended to locally free left *Poisson actions* (i.e., actions for which $\sigma: G \times P \rightarrow P$ is a Poisson map, where $G \times P$ is equipped with the product Poisson structure). In this case we obtain:

a) for all $g \in G$, $\sigma_g: P \rightarrow P$ is a pre-Poisson map.

b) the deficit distribution of σ_g is generated by the infinitesimal action of $\mathfrak{h}^g \subset \mathfrak{g}$.

If \mathfrak{h}^g is a Lie subalgebra of \mathfrak{g} and P/H^g is a smooth manifold, where H^g the connected subgroup of G integrating \mathfrak{h}^g , then P/H^g has a Poisson structure for which the projection map $\pi: P \rightarrow P/H^g$ is Poisson. This is a well-known fact, see [19, Thm. 6] or [17, Prop. 3.4]. Cor. A.6 in addition tells us that $\pi \circ \sigma_g: P \rightarrow P/H^g$ is also a Poisson map.

Remark A.9. Consider the action by left multiplication G on itself, and let $g \in G$ so that \mathfrak{h}^g is a Lie subalgebra of \mathfrak{g} . Then $H^g \backslash G$ (if smooth), together with the action of G by right multiplication, is a right Poisson homogeneous space (i.e., $(H^g \backslash G) \times G \rightarrow H^g \backslash G$ is a transitive right action and a Poisson map). Further both the projection π and $\pi \circ L_g: G \rightarrow H^g \backslash G$ are Poisson maps which are equivariant for the G -actions by right multiplication.

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